Level 3
The Secret of the Unicorn

Tintin kept on thinking about what was the hidden secret of the Unicorn that Barnaby wanted to tell. Until one day Snowy accidentally collides with the table on which the Unicorn was placed and the model falls on the floor. The broken model, from under its deck, reveals an ancient-looking paper, which reads:

"A great journey, awaits for thee;
For this piece of paper is a key
To riches and assets that know no bounds,
Buried somewhere, waiting to be found.
But first, I must take your measure,
And see if you're worthy of the ancient treasure.
Solve these puzzles, and they shall yield,
The location where the treasure lies, concealed."

Note: Three of the following puzzles claim to give the coordinates of the treasure. However, their answers may or may not be the same. Each question and its answer is independent of the other.

3.1 Problem 1

If the treasure is at a° North and b° East, then (a, b) are positive integers that satisfy $1 + 2^{2a+1} = b^2 - 2^a$. Determine all possible pairs (a, b).

Submission Guideline: Submit as (a_1, b_1),$(a_2, b_2)\cdots(a_n, b_n)$ where (a_i, b_i) are the valid pairs. If no such pair exists, submit 0.

3.2 Problem 2

If that was tough for you, try out this:
If the treasure is at x° North and y° East, then (x, y) are positive integers that satisfy $x^3 - x = y^7 - y^3$, and x is prime. Determine all possible pairs (x, y).

Submission Guideline: Submit as (x_1, y_1),$(x_2, y_2)\cdots(x_n, y_n)$ where (x_i, y_i) are the valid pairs. If no such pair exists, submit 0.

3.3 Problem 3

Still confused with the coordinates, how about this:
If the treasure is at n° North and r° East, then (n, r) be a pair of numbers with $r \in \mathbb{N}$ and a natural $n \geq 2$ that satisfy $L_n = r\Psi_n + 1$, where $L_n := \text{lcm}(1, 2, ..., 2n)$ and $\Psi_n := \binom{2n}{n}$. Determine all possible pairs (n, r).

Submission Guideline: Submit as (n_1, r_1),$(n_2, r_2)\cdots(n_n, r_n)$ where (n_i, r_i) are the valid pairs. If no such pair exists, submit 0.

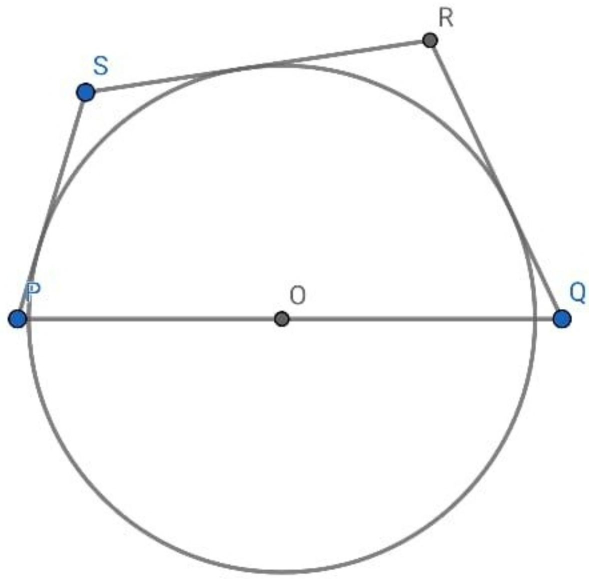
3.4 Problem 4

If you don't like coordinates, then let me tell you about the timezone.

If there be a circle such that it has centre on the side PQ of the cyclic quadrilateral $PQRS$ and the other three sides be tangent to the circle, then the treasure lies at a place in the timezone of GMT + $(PS + QR)^2$ hours.

Given: Length of PQ is given as 9.

<u>Submission Guideline</u>: Submit the value of $(PS + QR)^2$.



3.5 Problem 5

The treasure is kept neither too high, nor too low. Let,

$$p, q, r \in \mathbb{R}^+$$
$$p^2 + q^2 = 9$$
$$q^2 + qr + r^2 = 16$$
$$p^2 + \sqrt{3}pr + r^2 = 25$$

Then the height of the treasure is $(2pq + pr + \sqrt{3}qr)$ feet above sea level.

<u>Submission Guideline</u>: Submit the height of the treasure above sea level.