Level 4
Journey to the Treasure

With the coordinates for the treasure figured out, Tintin and Captain Haddock are ready to leave on their journey to reach the treasure.

4.1 Problem 1

Just as they were going to leave the hotel, they are visited by their old friend Prof. Calculus. Prof. Calculus pleads them to help him with an intriguing problem he has come across before they leave-

Let ABC be an acute triangle with circumcenter O such that $AB = 4$, $AC = 5$, $BC = 6$. Let D be the foot of the altitude from A to BC and E be the intersection of lines AO and BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY \parallel AO$ and $YO \perp AX$. Determine the length of BX.

Submission Guideline: Submit the length of BX. If it is a fraction, submit as a fraction, not in decimals.

4.2 Problem 2

Finally, the two of them leave the hotel, and come across an 8 by 8 yard farm. Tintin manages to cross the farm with ease, but the Captain, in his drunken state and having lost all sense of direction, is rooted to his spot.

Initially, Tintin and Captain are at the opposite corners of the farm. Tintin is standing at his spot, and waiting for Captain to make his way across to him.

In one minute, the captain can move i yards either horizontally (left or right both), or vertically (forward or backwards both) ($1 \leq i \leq 8$) (but not diagonally or in any other way). Assume that the probability of each move is equal, and the captain cannot leave the boundaries of the farm. What is the expected amount of time for which Tintin will have to wait?

Note on submission: If the expected time is x minutes, then submit x.

4.3 Problem 3

After travelling for about 80 days around the world, Tintin and Haddock finally reach the treasure coordinates. There they find a locked trunk with the inscribed message-

"Congratulations! Reaching the treasure isn't cakewalk
$20p^3 = q^3 + 1$ is the equation of the lock!
The pin is a possible value of (p, q) both of which should be prime,
If you aren't the heir of Haddock, then opening the lock would be a crime!"

Help Tintin and Haddock unlock the trunk by finding all prime numbers p and q that satisfy the equation of the lock.

Submission Guideline: Submit all possible (prime) values of (p, q) separated by commas.

4.4 Problem 4

Inside the trunk, there is a antique-looking box, which has a steering-wheel shaped lock. He saw an instruction manual stuck to the bottom of the box. The manual read:

"Let PQR be a triangle with $PQ = PR$. The angle bisectors of $\angle RPQ$ and $\angle PQR$ meet the sides QR and RP at S and T, respectively. Let I be the incenter of ΔPSR. Suppose that $\angle QTI = 45°$. In order to open the box, the steering wheel needs to be rotated by an angle equal to the sum of all possible values of $\angle RPQ$."

Help Tintin figure out the angle by which he should rotate the lock in order to open the box.

Submission Guideline: If he needs to rotate it by an angle of $x°$, then submit the value x.