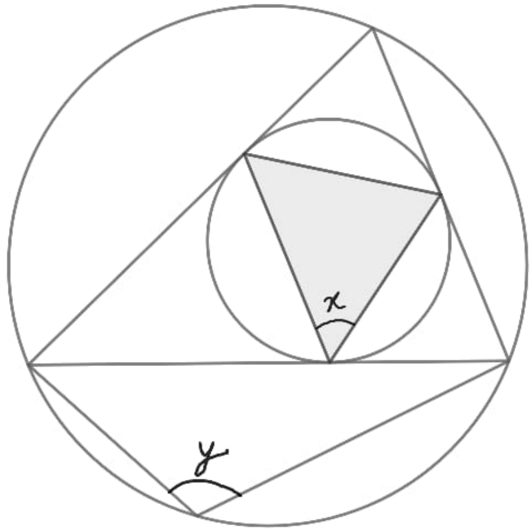
Solutions: Tintin and the Secret of the Unicorn

Level 1- Tintin Buys the Unicorn

Tintin and Snowy arrive in Newto(w)n (a beach town populated by geeky Math enthusiasts called Algebros) to meet their old friend Captain Haddock . With some time in his hand, Tintin decides to visit the market to buy a gift for Haddock.

1. Tintin looks up the market on the local directory of Newtown and finds the following:
   Welcome to Newtown
   To reach the Central Market, go along the direction of the tangent to the curve $y = f(x)$ at $x = 42$, where x and y are the angles shown in the figure.

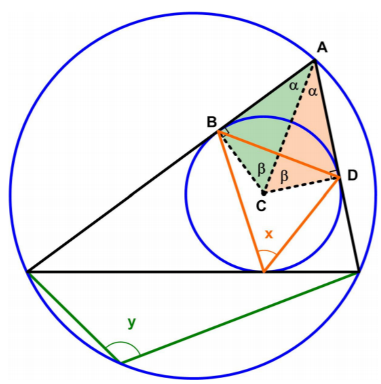
   Help Tintin figure out which direction he should go along.
   Note on submission: Submit the value of the slope of the said tangent .



▼ Solution :

Answer = 2

$y = 2x$



Then we have:

$$\alpha + \beta = 90$$
$$x = (1/2)2\beta = \beta$$
$$y + 2\alpha = 360/2 \Rightarrow \alpha = 90 – y/2$$
$$\Rightarrow 90 = \alpha + \beta = (90 – y/2) + x$$
$$\Rightarrow y = 2x$$

2. *On his way to the Central Market, Tintin comes to a fork on the road. With two roads in front of him, he is confused about which one leads to the Central Market. He decides to ask an Algebro(a native of Newtown) the correct way.* The Algebro replies:

> Oh young traveler, you seem astray.
> But I can show you the right way.
> One question is all you've got,
> But I might lie, or I might not.

Help Tintin in framing the one question he can ask the Algebro, such that irrespective of whether the Algebro lies or not, Tintin would know which way to go.

Submission Guideline: Submit the question you want to ask the stranger.

▼ Solution :

The challenge here is to find a question that forces a liar to lie about a lie and hence tell the truth. There can be a number of valid variations but the basic idea would be the same.

This works: point at one of the forks and ask the native: “If I were to ask you if this road leads to the market, would you say yes?”

If the fork is the correct one, a liar would answer no to the direct question ‘does this road lead to the market?’ and thus his (lying) answer to the actual question must be yes.

The truth-teller will also answer yes if the road is the correct one.

3. *On reaching the market, Tintin realises that he needs Eulors (currency of Newtown) to buy a gift. So he visits the local bank and requests for a loan. The banker tells him the following:*

   I have 3 bags here: X, Y and Z. Bag X has 27 one Eulor coins and bags Y and Z are empty. If you tell me the minimum number of steps (if possible) required to get 9 coins in each bag, you can take all the money. But there are two rules:

   - In the jth step, you have to shift exactly j coins from one bag to another.
   - You cannot transfer any coin directly between Y and Z.

   Submission guideline: Submit 0 if not possible. Else submit the minimum number of steps required.

   ▼ Solution :

   To Y: +2, +7
   To Z: +1, +3, +4, 5, +6
   Hence the Answer is: 7

4. *Stocked up with 27 eulors, Tintin starts shopping. He visits the jewellery shop Tu-ring and the mobile shop, Des-Carts, however, finds nothing interesting. Just then, he spots a*

model of the Unicorn, the 15th century warship captained by Haddock's ancestors. Convinced that he had found just the perfect gift for Haddock, Tintin asks the shopkeeper for the price of the Unicorn model. The shopkeeper tells Tintin the following:

Initially I had 5 boxes of ship-models, consisting of three types of ships: the tanker, the steamer and the cruise. Each box had the same number of ships in them. One day, I sold all of it to 8 traders. Each trader bought the same number of ships, paying 17 Eulors for each cruise, 4 Eulors for each steamer and 2 Eulors for each tanker. In the end, I had 301 Eulors in total.
Then I bought the Unicorn- it cost me all the money I had earned from selling, all the tankers combined. That, will be the price for you as well!
Assume that the shopkeeper had the maximum number of ships possible with the given constraints, and he had at-least 2 model ships of each kind. Find out the price of the Unicorn (i.e. the money earned by sale of all the tankers by the shopkeeper)

Submission Guideline: If the price of the Unicorn is x Eulors, submit x.

▼ Solution :

As there were five boxes with an equal number of ships in each box, the total number of ships must be divisible by 5; and as every one of the eight traders bought the same number of ships, the number must be divisible by 8. Therefore the number must be a multiple of 40. The highest possible multiple of 40 that will work will be found to be 120, and this number could be made up in one of two ways–1 cruise, 23 steamers, and 96 tankers, or 3 cruise, 8 steamers, and 109 tankers. But the first is excluded by the statement that he had at-least 2 model ships of each kind. Therefore the second grouping is the correct answer. So, the price of the ship is 109*2=218.

$x = 218.$

Note: A number of participants said that the 27 Eulors in the beginning confused them. Sorry if it was confusing, however, the question asked for the cost of the unicorn, and hence, it is nor important in this question if it is greater or lesser than 27 Eulors.

5. *Just as Tintin was exiting the shop with the Unicorn model, a man named Ivanovich approaches Tintin saying that the Unicorn model he had just bought was incredibly important for him and offered Tintin a billion Eulors, for the Unicorn. Tintin denies the offer. After many more trials, desperate Ivanovich says the following:*

   "I offer you a deal. There will be 3 heaps of sticks, consisting say x, y and z sticks respectively (x, y and z may or may not be equal). Each one of us will take turns alternatively and in each turn, the person can remove any number of sticks from any one heap. If I pick up the last stick, I lose and I'll give you 100 thousands Eulors. However, if you pick up the last stick, you lose and you give me the Unicorn for free. I have with me

30 sticks in all. You can form the heaps as you want (determine x, y and z such that there sum is 30) however, I will take the first turn. Is that a deal?"

Tintin thinks about it for a moment and then, agrees to the deal since he knows that there exists some set of values (x, y, z) such that he will definitely earn the 100 thousand dollars, irrespective of how Ivanovich plays.

Submit any set of values of x, y and z (the number of sticks in each heap) such that if Ivanovich takes the first chance in the game, Tintin would definitely win, assuming Tintin plays correctly.

Submission Guideline: Submit the ordered set (x, y, z)

▼ Solution :

Any of these combinations would win:

(15,14, 1); (15, 13, 2); (15, 12, 3); (15, 11, 4); (15, 10, 5); (15, 9, 6); (15, 8, 7); (14, 13, 3); (14, 11, 5); (14, 9, 7); (13, 11, 6); (13, 10, 7); (12, 11, 7)

Solution: basic strategy for game of Nim

The nim is a solved combinatorial game, which means that one player can guarantee a win. The way to win a nim is as follows:

1. Convert the heap sizes into binary.
2. Calculate the digital sum, that is the binary sum neglecting all carries from a digit to another. This is known as the nim-sum.
3. Finish every move so that the nim-sum equals zero.

An interesting fact is that if the nim-sum is currently zero, then the player cannot remove objects to make the nim-sum equal to zero again. In contrast, if the nim-sum is currently not zero, then the player can always finish the move to make the nim-sum zero. Therefore if the initial nim-sum is zero, then the player who goes second can win no matter what. If the initial nim-sum is not zero, then the player who goes first can gurantee win.

Note: Some teams submitted a solution with 0 sticks in a heap. This was not accepted because the question clearly asks you to make 3 heaps. 0 sticks in a heap means you are making 2 heaps.

Level 2- Barnaby's Assassination

After winning the game against Ivanovich, Tintin finally gets the Unicorn to his hotel, Sri Nivasa. Soon after, he gets a call from a man named Barnaby. Barnaby claims that the model of the Unicorn has a secret hidden in it and that Ivanovich has planned an attempt to steal the Unicorn. He requests a meeting with Tintin. Curious to know the reason behind Ivanovich's desperation to get the Unicorn, Tintin agrees to meet Barnaby. The doorbell rings at the pre-decided time, but as soon as Tintin opens his door, a gun shot is fired on Barnaby and he falls dead at Tintin's door. The gunman who was in the corridor runs away.

1. To investigate the curious incidents, Tintin calls up his detective friends Thompson and Thomson. As they meet up, they have the following conversation:

   Tintin : How many criminals have you caught till date?

   Thomson: I can tell you but I won't tell Thompson.

   Thompson: So can I and neither would I tell Thomson.

   So both of them separately whisper the number of criminals each has caught in Tintin's ear, careful that the other doesn't listen.

   Tintin: Both of you have caught at least one criminal but one of you has caught one more than the other.

   Thomson: I have no idea if you have caught more criminals than I did.

   Thompson: Me neither. Do you know now?

   Thomson: Yes, indeed!

   Thompson: Really? Then so do I!

   Note: They have not caught any criminal together (i.e. no criminal is common for both of them)

   Submission Guideline- If there are p possible values for the number of criminals Thomson has caught, then **submit the sum of all the p values**. If $p = 0$ (i.e. no possible case), then submit 0.

   ▼ Solution :

   Answer: 5 (2 and 3)

   NOTE: Most teams submitted the answer 2. We have accepted the answer 2 as well due to a miscommunication on our side. However, the technically correct answer is 5.

   If someone had caught only one criminal, then that person would know the answer immediately. Since both Thompson and Thomson did not know the answer initially, both must have caught at least 2.

   Let us assume that Thomson caught 2 criminals, and analyze the conversation in Thomson's perspective. Thomson knows that Thompson caught either 1 or 3 criminals. Thus initially Thomson would not know if he caught more criminals than Thompson did. However, Thompson said "me neither", which implies that Thompson did not catch just 1 criminal. Hence Thompson must have caught 3 criminals, which Thomson now knows with certainty. So, Thomson could reply "yes, indeed!".

This time suppose that Thomson caught 3 criminals. Then Thomson knows that Thompson caught either 2 or 4 criminals. Again, Thomson would not know if he caught more criminals than Thompson did. Now, assume that Thomson caught two criminals. Then Thompson initially knows that Thomson caught either 1 or 3. After listening to Thomson's answer "I have no idea", Thompson would then know that Thomson had caught 3 criminals. However, as Thompson replied "me neither", Thompson could not have caught two criminals. Therefore Thomson knows that Thompson has caught 4 criminals.

Next, assume that Thomson caught 4 criminals. Initially Thomson knows that Thompson caught either 3 or 5 criminals. At this point Thomson does not know if he caught more criminals than Thompson did. Now, let's say that Thomson caught 3 criminals. Then Thompson initially knows that Thomson caught either two or four criminals. In both cases, Thomson would not know if he has caught more criminals than Thompson; thus Thomson's reply "I have no idea" does not yet give additional information to Thompson. Thus Thompson replies "me neither". Thomson also knows that Thompson would not know the answer yet, whether Thomson had watched Thompson 3 or 5 times. Hence Thomson also does not know the answer at this point, and therefore Thomson wouldn't have replied "yes, indeed!". Therefore Thomson did not catch four criminals.

In a similar manner, if Thomson had caught more than 4 criminals , then it would require a longer conversation for Thomson and Thompson to figure out the answer.

In conclusion, Thomson could have caught either 2 or 3 criminals..

2. Done with the small talk, Thompson and Thomson finally begin the investigation. Initially, all the n people who live in the locality are considered equally likely to have shot Barnaby. However, the old lady from the adjacent room (also an eyewitness to the entire incident) reports that the assassin had 6 fingers on his right hand.

   Let x and y be the respective probabilities that an innocent man has six fingers on his right hand and that the assassin has six fingers on his right hand. Note that $x < y$ and y may be less than 1 as old eye witnesses are not entirely reliable.

   It is found that Ivanovich, who lives in the locality, has 6 fingers in his right hand. What is the probability that Ivanovich is the assassin?

   Note: Thompson and Thomson have no prior knowledge about Ivanovich and his curiosity in the ship. Hence, this being a Math question, does not take into consideration any qualitative premonitions about Ivanovich

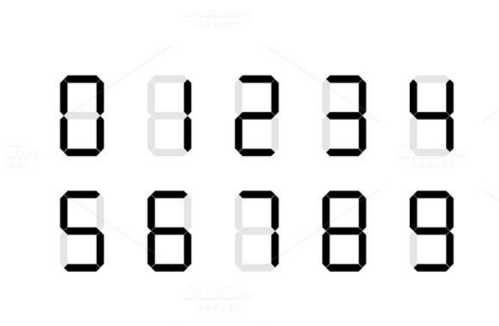
   Submission Guideline: Submit the expression in terms of x and y. Please use brackets extensively, as any confusion in syntax would lead to rejection of solution.

   ▼ Solution :

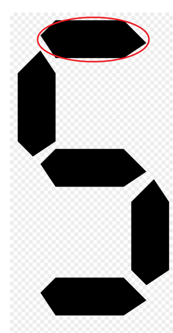
Let R be the event that Ivanovich is guilty and M be the event that he has six fingers on his right hand. Let $a = x/y$. By Bayes' rule and Law of Total Probability,

$$P(R|M) = \frac{P(M|R)P(R)}{P(M|R)P(R) + P(M|R^c)P(R^c)} = \frac{y \cdot \frac{1}{n}}{y \cdot \frac{1}{n} + x(1 - \frac{1}{n})} = \frac{1}{1 + a(n-1)}$$

3. Shaken by the killing of Barnaby, Tintin decides to design a digital lock for protecting the Unicorn. The digital lock has a display which uses digital numbers of the kind shown below-

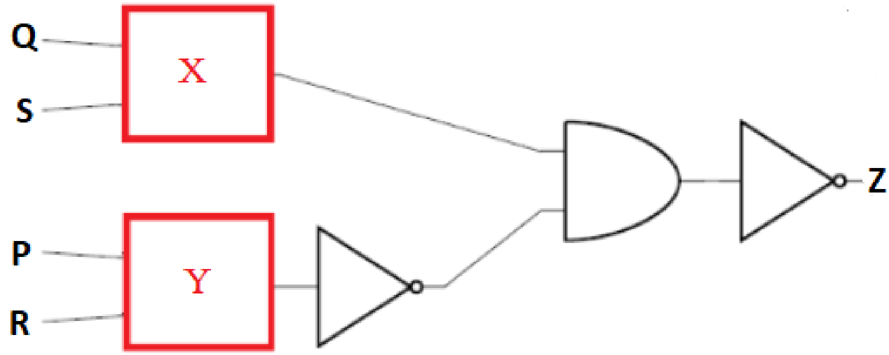


While constructing the display, he got confused about the logic gates leading to this portion of the circuit marked here-

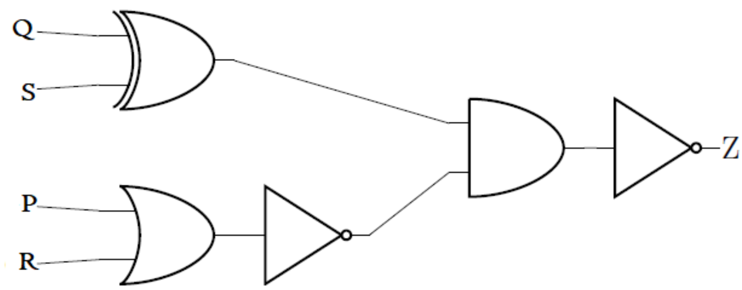


He has designed the following circuit of logic gates. The decimal digit to be represented is first converted into binary, with 4 binary digits P, Q, R, S. These are the input to the circuit. Can you help Tintin complete the circuit by determing which logic gate would take place of the boxes labelled X and Y.

Submission guideline: Submit answer as (X,Y) with X and Y as the names of the logic gate.



▼ Solution :



We want every digit **except** 1 and 4 to pass through with a 1.

Since there is a NOT gate at the very end, that means we want the AND to output a 1 if the digit **is** 1 or 4.

1 is 0001 in binary and 4 is 0100 in binary. So we need two checks:

Check 1: Q and S are different.

Check 2: P and R are both 0.

Q XOR S will return 1 if Q and S are different.

NOT (P OR R) will return 1 if P and R are both 0.

So we putting both parts together...

(Q XOR S) AND NOT (P OR Q)

... will return 1 if the digit is a 1 or a 4. That means to get a 1 when the digit is something else, we just run a NOT gate:

NOT ((Q XOR S) AND NOT (P OR R))

Note : (NOR, XNOR) is also correct

4. To further protect the Unicorn, Tintin decides to setup a laser security system in his hotel courtyard.

   The courtyard is a $10 \text{ metres} \cdot 10 \text{ metres}$ square, covered with 100 square tiles each of $1 \text{ metre} \cdot 1 \text{ metre}$ A single laser emitter occupies one tile and emits lasers along the two diagonals of the square (as shown in fig 2a). Hence, if there is any obstruction on any tile located on the diagonals of the laser-emitter-tile, an alarm would be triggered. What are the maximum number of laser emitters that Tintin can place in this courtyard?

   (Note: No laser emitter may be placed on the tiles secured by another laser emitter as that would be considered as an obstruction and would perpetually buzz the alarm. However, a tile may be secured by multiple laser emitters.)

   Submission guideline- Submit the maximum number of laser emitters

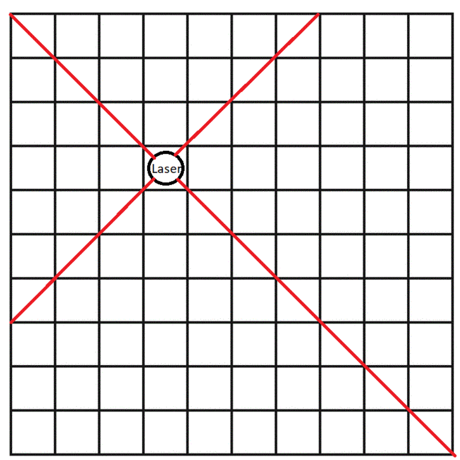


Fig 2a

▼ Solution :

Answer: 18

NOTE: A lot of people submitted the correct answer and in the solution, wrote about how they got to this number. However, extremely few people proved that why 18 is the maximum number. Here is the complete proof.

This question is equal to the popular problem of placing maximum bishops on a chessboard without any bishop threatening the other. Solution to that problem is 14 as explained below. We just extended it to a 10*10 board.

Proof for maximum in the bishop problem.

Let the chessboard be labelled from (1,1)(1,1) to (8,8)(8,8).

It is obvious that we can get an arrangement of 14 bishops. E.g. Having them in the top and bottom most row of all the columns except the right most.

It remains to show that 14 is indeed an upper bound. Such an argument typically arises from the pigeonhole principle, but it might not be obvious how to do so.

Observe that if two bishops can attack each other on the same top-left, bottom-right diagonal, then the sum of their coordinates is the same. As such, let's create the following pigeonholes:

- PH 1: Coordinates sum to 3.
- PH 2: Coordinates sum to 4.
- PH 3: Coordinates sum to 5.
- PH 4: Coordinates sum to 6.
- PH 5: Coordinates sum to 7.
- PH 6: Coordinates sum to 8.
- PH 7: Coordinates sum to 9.
- PH 8: Coordinates sum to 10.
- PH 9: Coordinates sum to 11.
- PH 10: Coordinates sum to 12.
- PH 11: Coordinates sum to 13.
- PH 12: Coordinates sum to 14.
- PH 13: Coordinates sum to 15.

We're still missing 2 squares, namely (1,1)(1,1) and (8,8)(8,8), which we will place in the same pigeonhole:

- PH 14: Squares (1,1) and (8.8).

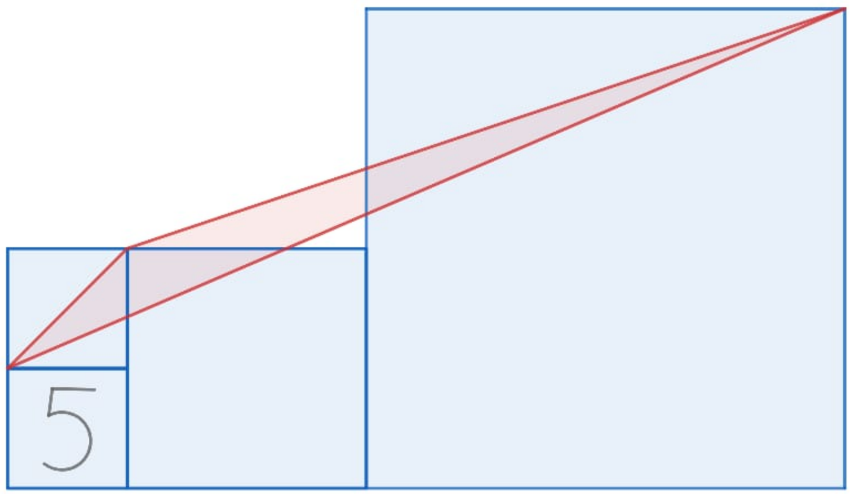
  (1,1)

  (8.8)

Then, if we're given 15 or more bishops, we are guaranteed that there are at least $\lceil \frac{15}{14} \rceil = 2$ bishops that are in the same pigeonhole. It is clear that these 2 will attack each other. Hence, 14 is indeed an upper bound.

Since 14 can be achieved and is an upper bound, thus it is the maximum.

5. Tintin then decides to step up the security of the hotel lobby, by installing landmines. He comes up with a floor-plan to lay down the mines. The floor-plan looks as follows: (redraw the diagram)



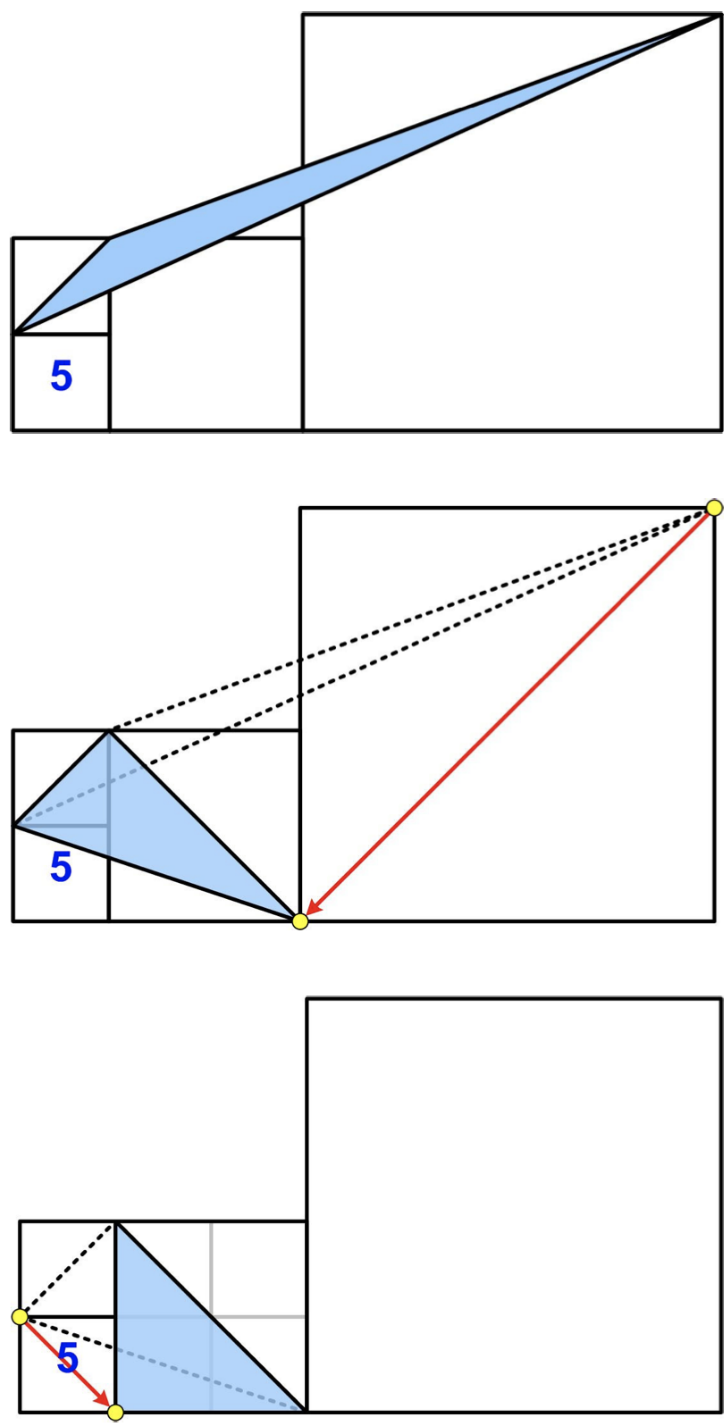
Tintin knows that the area of his room (the bottom left square is 5 meter squares. If the cost of laying down a mine is 10 Eulor/meter-square, what is the amount Tintin will have to spend in order to mine the red area?

Submission Guideline- **Submit the amount required, along with explanation.**

▼ Solution

Area is 10, so the cost will be 100.

Sheering the triangle by moving its top vertex parallel to its base doesn't change the area, since the altitude remains constant.



Level 3- The Secret of the Unicorn

Tintin kept on thinking about what was the hidden secret of the Unicorn that Barnaby wanted to tell. Until one day Snowy accidentally collides with the table on which the Unicorn was placed and the model falls on the floor. The broken model, from under its deck, reveals an ancient-looking paper, which reads:

"A great journey, awaits for thee;
For this piece of paper is a key
To riches and assets that know no bounds,
Buried somewhere, waiting to be found.
But first, I must take your measure,
And see if you're worthy of the ancient treasure.
Solve these puzzles, and they shall yield,
The location where the treasure lies, concealed."

Note: Three of the following puzzles claim to give the coordinates of the treasure. However, their answers may or may not be the same. Each question and its answer is independent of the other

1. If the treasure is at a° North and b° East, then (a,b) are positive integers that satisfy $1+2^{2a+1}=b^2-2^a$. Determine all possible pairs (a,b).
   Submission Guideline: Submit as $(a_1,b_1),(a_2,b_2)\cdots(a_n,b_n)$ where (a_i,b_i) are the valid pairs. If no such pair exists, submit 0.

   ▼ Solution :

   $(a,b)=(4,23)$

   $$\begin{aligned} &1+2^a+2^{2a+1}=b^2 \\ \implies\ &2^a(2^{a+1}+1)=(b+1)(b-1) \end{aligned}$$

   $a<-1$ gives a non-integer LHS (no solutions)

   $a=-1$ gives LHS = 1 with no solutions for b

   $a=0$ gives LHS = 3 and $b=\pm 2$

   For $a>0$, b is odd so put $b=2k+1$ and $2^a(2^{a+1}+1)=(2k+2)(2k)=4k(k+1)$ which is divisible by 8 so $a\geq 3$ and $2^{a-2}(2^{a+1}+1)=k(k+1)$.

   Clearly we cannot have $k=2^{a-2}$ or $k+1=2^{a-2}$ so we need $(2^{a+1}+1)$ to split into (odd) factors r,s such that $2^{a-2}r=s\pm 1$.

   Then $|r|<8$ otherwise $2^{a-2}|r|>(2^{a+1}+1)$. Also $2^{a-2}r^2=sr\pm r$ so $|x|=2$ is the only viable choice and $2^{a-2}9=(2^{a+1}+1)\pm 3$ gives $2^{a-2}=4$ i.e. $a=4$ (and $b=\pm 23$) as the only other solution.

   In summary: the *only* solution (a,b) is $(4,23)$ after applying the condition.

2. *If that was tough for you, try out this:*

If the treasure is at x° North and y° East, then (x, y) are positive integers that satisfy $x^3 - x = y^7 - y^3$, and x is prime. Determine all possible pairs (x, y)

Submission Guideline: Submit as $(x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)$ where (x_i, y_i) are the valid pairs. If no such pair exists, submit 0.

- ▼ Solution :

  $(x, y) = (5, 2)$

  The given equation is $x(x-1)(x+1) = y^3(y^2+1)(y+1)(y-1)$. As the factor p on the LHS is a prime, it must divide one of the factors $y-1, y, y+1, y^2+1$ on the RHS.

  A key point to deduce is that $x > y^2$. One way to do this is as follows. The LHS $=$ $x^3 - x$ is an increasing function of p for $x \geq 1$, e.g. because the derivative $3x^2 - 1$ is positive. So for any given $y \geq 1$, there is exactly one real value of x for which LHS = RHS. Trying $x = y^2$ gives LHS $= y^6 - y^2 < y^7 - y^3 =$ RHS, e.g. because $y^7 - y^3 - (y^6 - y^2) = (y^6 - y^2)(y-1) > 0$.

  As the prime x is greater than y^2, it cannot divide any of $y-1, y, y+1$. So x must divide y^2+1 and therefore we must have $x = y^2+1$, again because $x > y^2$. Substituting this in the given equation, we get $(y^2+1)y^2(y^2+2) = y^3(y^2+1)(y+1)(y-1)$. Cancelling common factors gives $y^2+2 = y^3 - y$, i.e. $2 = y^3 - y^2 - y$. This has a unique integer solution $n = 2$, e.g. because the factor y on the RHS must divide 2 and now one checks that $y = 2$ works. So $y = 2$ and the prime $x = y^2 + 1 = 5$ give a unique solution to the given equation.

3. Still confused with the coordinates, how about this:

   If the treasure is at n° North and r° East, then (n, r) be a pair of numbers with a natural $n \geq 2$ that satisfy $L_n = r\Psi_n + 1$, where $L_n := \operatorname{lcm}(1, 2, ..., 2n)$ and $\Psi_n := \binom{2n}{n}$. Determine all possible pairs (n, r)

   Submission Guideline: Submit as $(n_1, r_1), (n_2, r_2) \cdots (n_n, r_n)$ where (n_i, r_i) are the valid pairs. If no such pair exists, submit 0.

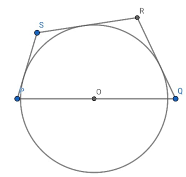
   - ▼ Solution

     No such pair exists.
     The n^{th} Catalan Number $C_n = \frac{1}{n+1} \cdot \binom{2n}{n}$ is a natural number, so $n+1 | \binom{2n}{n}$.
     Also, $n+1 | \operatorname{lcm}(1, 2, ..., 2n)$. Hence, no such pair exists.

4. If you don't like coordinates, then let me tell you about the timezone.

   If there be a circle such that it has centre on the side PQ of the cyclic quadrilateral $PQRS$ and the other three sides be tangent to the circle, then the treasure lies at a place in the timezone of GMT + $(PS + QR)^2$ hours.

Given: Length of PQ is given as 9.
Submission Guideline: Submit the value of $(PS + QR)^2$



▼ Solution :

$(PS + QR)^2 = PQ^2 = 81$

'''A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Proof: Let O be the center of the circle mentioned in the problem. Let T be the second intersection of the circumcircle of CDO with AB. By measures of arcs, $\angle DTA = \angle DCO = \frac{\angle DCB}{2} = \frac{\pi}{2} - \frac{\angle DAB}{2}$. It follows that $AT = AD$. Likewise, $TB = BC$, so $AD + BC = AB$, as desired. '''

And now, this result can be directly used to get the answer.

5. The treasure is kept neither too high, nor too low.

   Let
   $p, q, r \in \mathbb{R}^+$
   $p^2 + q^2 = 9$
   $q^2 + qr + r^2 = 16$
   $p^2 + \sqrt{3}pr + r^2 = 25$

   Then the height of the treasure is $(2pq + pr + \sqrt{3}qr)$ feet above sea level.

   Submission Guideline: Submit the height of the treasure above sea level

   ▼ Solution :

   24

   Consider the right angled triangle ABC with sides 3, 4, 5 and an interior point O such that $AO = x$, $\angle AOB = 90$ and $CO = z$, $\angle COA = 150$ and $BO = y$, $\angle BOC =$

120. Then the three given equations are in fact cosine rule for each of the triangle prescribed above. For example, in ΔBOC we have

$$\begin{aligned} 4^2 &= y^2 + z^2 - 2yz\cos(120) \\ &= \quad y^2 + z^2 + yz \end{aligned}$$

The area of ΔABC (which is 6) calculated using the sine formula (for each of the smaller triangle) gives us

$$6 = \frac{1}{2}xy + \frac{1}{2}yz\,\sin 60 + \frac{1}{2}\,\sin 30$$

So, the answer is 24.

Level 4- Journey to the Treasure

With the coordinates for the treasure figured out, Tintin and Captain Haddock are ready to leave on their journey to reach the treasure.

1. *Just as they were going to leave the hotel, they are visited by their old friend Prof. Calculus. Prof. Calculus pleads them to help him with an intriguing problem he has come across before they leave-*

   Let ABC be an acute triangle with circumcenter O such that $AB = 4, AC = 5, BC = 6$. Let D be the foot of the altitude from A to BC and E be the intersection of lines AO and BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY \,||\, AO$ and $YO \perp AX$. Determine the length of BX.
   Can you help Prof. Calculus find the length of BX.

   Submission Guideline: Submit the length of BX. If it is a fraction, submit as a fraction, not in decimals.

   ▼ Solution :

   $BX = 96/41$

   First we need to compute AD. We can do this by using the area of ABC (obtained from Heron's formula); compute

   $$AD = \frac{2[ABC]}{BC} = \frac{2}{6}\cdot\sqrt{\frac{15}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}} = \frac{1}{3}\cdot\frac{15}{4}\sqrt{7} = \frac{5}{4}\sqrt{7}$$

   This makes $BD = \sqrt{4^2 - \frac{25}{16}\cdot 7} = \frac{9}{4}$ and subsequently $CD = 6 - \frac{9}{4} = \frac{15}{4}$. So we set

$$D = (0,0)$$
$$B = (-9,0)$$
$$C = (15,0)$$
$$A = (0, 5\sqrt{7})$$

Here we are scaling coordinate system up by a factor of four to ease computation (by eliminating fractions).

Next, we ought to compute O. We can compute the circumradius using

$$\frac{abc}{4R} = \frac{15}{4}\sqrt{7} \implies R = \frac{8}{\sqrt{7}}$$

So the distance from O to BC is

$$\sqrt{\frac{8^2}{7} - 3^2} = \frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}$$

Also, noticing that O is directly overhead the midpoint of $\overline{BC}$, we can compute

$$o = (3, \frac{4}{7}\sqrt{7})$$

in out coordinate system. (The extra factor of four again comes from our scaling.)

Next we need to compute E. We can do so by finding x-intercept of the line AO. The slope of line AO is

$$\frac{5\sqrt{7} - \frac{4}{7}\sqrt{7}}{0-3} = -\frac{31}{21}\sqrt{7}$$

and hence the coordinate of E is

$$E = (\frac{5\sqrt{7}}{\frac{31}{21}\sqrt{7}}, 0) = (\frac{105}{31}, 0)$$

Now for a trick—we can encode the parallel condition by letting r denote the ratio between lengths of $\overline{XY}$ and $\overline{AE}$. Therefore

$$X = (\frac{105}{31}r, 0) \text{ and } y = (0, 5\sqrt{7} \cdot r)$$

(Similar triangles) Now the condition $\overline{AX} \perp \overline{YO}$ is just a slope condition. We hav

$$-1 = (\text{slope of } \overline{AX}) \cdot (\text{slope of } \overline{YO})$$

$$
\begin{aligned}
&= \frac{5\sqrt{7}-0}{0-\frac{105}{31}r} \cdot \frac{\frac{4}{7}\sqrt{7}-5\sqrt{7}\cdot r}{3-0} \\
&= \frac{-31}{21r} \cdot \frac{4-35r}{3} \\
\implies & \frac{21r}{31} = \frac{4-35r}{3} \\
\implies & 63r = 124 - 1085r \\
\implies & r = \frac{31}{287}
\end{aligned}
$$

We are home free—note that

$$X = (\frac{105}{31} \cdot \frac{31}{287}, 0) = (\frac{15}{41}, 0)$$

Hence, subtracting and scaling back gives

$$BX = \frac{1}{4}\left(\frac{15}{41} + 9\right) = \frac{96}{41}$$

and we are done.

2. *Finally, the two of them leave the hotel, and come across an 8 by 8 yard farm. Tintin manages to cross the farm with ease, but the Captain, in his drunken state and having lost all sense of direction, is rooted to his spot.*
Initially, Tintin and Captain are at the opposite corners of the farm. Tintin is standing at his spot, and waiting for Captain to make his way across to him.
In one minute, the captain can move i yards either horizontally (left or right both), or vertically (forward or backwards both) $(1 \leq i \leq 8)$ (but not diagonally or in any other way). Assume that the probability of each move is equal, and the captain cannot leave the boundaries of the farm.
What is the expected amount of time for which Tintin will have to wait?

Note on submission: If the expected time is x minutes, then submit x

▼ Solution :
$x = 70$

This problem is equivalent to a chess problem, with captain haddock playing the role of a rook.

There are three types of square on the chessboard. Type 1 squares are those from which the rook must take at least two moves to get the top-right corner, these are squares in rows 1-7 and in columns 1-7. Type 2 square are those from which the rook can reach the top-right target square in row 8 and column 1-7, or else in column 8 and rows 1-7.The only Type 3 square is the top-right target square in row 8 and column 8. We are counting rows from the bottom, and columns from the left.

From a Type 1 square, the rook will move to another Type 1 square with probability $\frac{6}{7}$, and will move to a Type 2 square with probability $\frac{1}{7}$. From a Type 2 square, the rook will move to a Type 1 square with probability $\frac{1}{2}$, will move to another Type 2 square with probability $\frac{3}{7}$, and will move to Type 3 square, ending the game, with probability $\frac{1}{14}$.

Given these observations, the expected number of moves until the game end only depends on the Type of the starting square, and not more generally on the actual square. Let E_1 be the expected number of moves to end the game, starting from a Type 1 square, and let E_2 be the expected number of moves to end the game, starting from a Type 2 square. Conditional expectations arguments tell us that

$$E_1 = \frac{6}{7}(E_1 + 1) + \frac{1}{7}(E_2 + 1)$$
$$E_2 = \frac{1}{2}(E_1 + 1) + \frac{3}{7}(E_2 + 1) + \frac{1}{14} \times 1$$

and hence

$$E_1 = E_2 + 7 \qquad\qquad 8E_2 = 7E_1 + 14$$

Solving these simultaneous equations yields $E_2 = 63$ and $E_1 = 70$.

3. *After travelling for about 80 days around the world, Tintin and Haddock finally reach the treasure coordinates. There they find a locked trunk with the inscribed message-*

   Congratulations! Reaching the treasure isn't cakewalk

   $20p^3 = q^3 + 1$ is the equation of the lock!

   The pin is a possible value of (p, q) both of which should be prime,

   If you aren't the heir of Haddock, then opening the lock would be a crime!

   Help Tintin and Haddock unlock the trunk by finding all prime numbers p and q that satisfy the equation of the lock.

   Submission Guideline: Submit all possible (prime) values of (p, q) separated by commas.

   ▼ Solution :

   $(p, q) = (7, 19)$

Rewrite the equation as $20p^3 = q^3 + 1 = (q+1)(q^2 - q + 1)$. Since $20 = 4 \cdot 5$ must divide RHS, we can see that $q \equiv -1 \pmod 4$ and $q \equiv -1 \pmod 5$ that is to say, $q \equiv -1 \pmod{20} \Rightarrow q = 20k - 1$ for some positive integer k.
Plugging this back in, we get $p^3 = k(q^2 - q + 1)$, thus since a is prime, we either have $k = 1 \Rightarrow (p, q) = (7, 19)$ or $p = k \Rightarrow k^2 = (20k-1)^2 - (20k-1) + 1$ that however has no real solutions.
Thus the only pair is $(p, q) = (7, 19)$

4. *Inside the trunk, there is a antique-looking box, which has a steering-wheel shaped lock. He saw an instruction manual stuck to the bottom of the box. The manual read*:
"Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^{\circ}$. In order to open the box, the steering wheel needs to be rotated by an angle equal to the sum of all possible values of $\angle CAB$."
Help Tintin figure out the angle by which he should rotate the lock in order to open the box.

Submission Guideline: If he needs to rotate it by an angle of x°, then submit the value x

▼ Solution :

Possible values of $\angle RPQ = 60^{\circ} \text{ and } 90^{\circ}$

Note $\angle BCI = \alpha$. So $\angle CIE = 2\alpha$ and $\angle DIK = 90 - \alpha$.

By sine law in ΔEIK, we have: $\dfrac{IK}{\sin 45} = \dfrac{EK}{\sin 2\alpha}$

By sine law in ΔDIK, we have: $\dfrac{IK}{\sin 45} = \dfrac{DK}{\cos \alpha}$

So, $\dfrac{EK}{DK} = 2 \sin \alpha \cdots (1)$

$\angle KEA = 3\alpha + 45$. So, $\sin \angle KEC = \sin \angle KEA = sin(3\alpha + 45) = sin45(sin3\alpha + cos3\alpha)$

Sine law in ΔCEK implies $\dfrac{EK}{sin\alpha} = \dfrac{KC}{sin45(sin3\alpha + cos3\alpha)}$

Sine law in ΔCKD, we have $\dfrac{DK}{\sin \alpha} = \dfrac{KC}{\sin 45}$.

So, $KC = \dfrac{EKsin45(sin3\alpha + cos3\alpha)}{sin\alpha} = \dfrac{DKsin45}{sin\alpha}$.

So $\dfrac{DK}{EK} = sin3\alpha + cos3\alpha$. Multiplying this equation with (1), we get:

$1 = 2sin\alpha(sin3\alpha + cos3\alpha)$

Transforming product to sum we get:

$1 = cos2\alpha - cos4\alpha - sin2\alpha + sin4\alpha$

$1 = cos2\alpha - 2cos^2 2\alpha + 1 - sin2\alpha + 2sin2\alpha cos2\alpha$

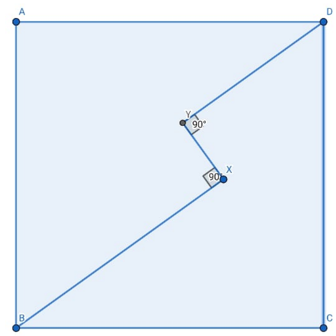
Factorizing the equation we get:

$(2cos2\alpha - 1)(sin2\alpha - cos2\alpha) = 0$

So, $\angle BAC = 60 \text{ or } 90.$

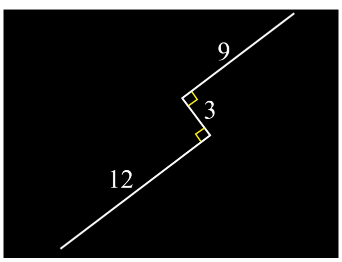
Wild Card

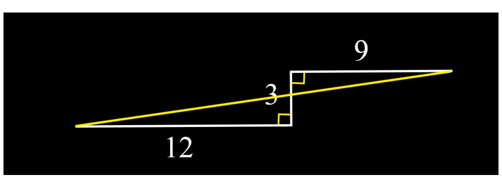
1. Let $ABCD$ be a square and BX and DY be two parallel lines such that $XY \perp BX$. Also, $BX = 12$, $XY = 3 \text{ and } DY = 9.$
   Compute the side length of the square $ABCD$.

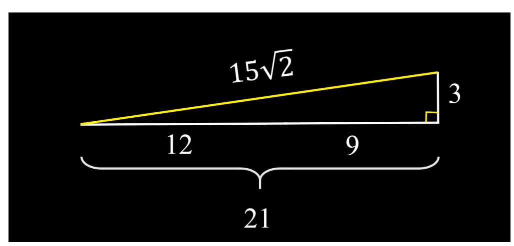


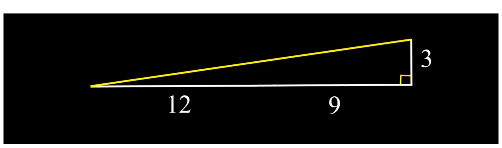
▼ Solution :

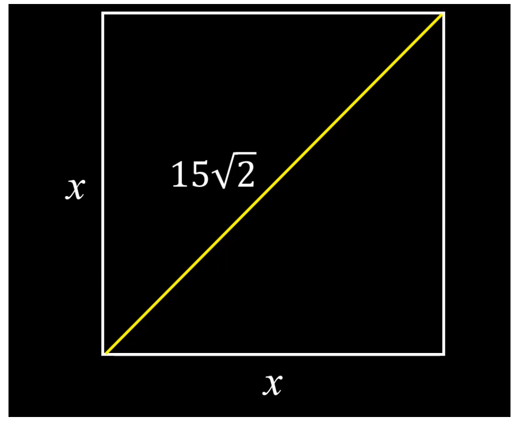
$x = 15$











2. Compute
$\sum_{1 \le x < y < z} \frac{1}{2^x 3^y 5^z}$, $x, y, z \in \mathbb{N}$

▼ Solution :

$1/1624$

Let $x = b - a$ and $y = c - b$ so that $c = a + x + y$. Then

$$2^a 3^b 5^c = 2^a 3^{a+x} 5^{a+x+y} = 30^a 15^x 5^y$$

and a, x, y are any positive integers. Thus

$$\begin{aligned} \sum_{1 \le a \le b \le c} \frac{1}{2^a 3^b 5^c} &= \sum_{1 \le a, x, y} \frac{1}{30^a 15^x 5^y} \\ &= \sum_{1 \le a} \frac{1}{30^a} \sum_{1 \le x} \frac{1}{15^x} \sum_{1 \le y} \frac{1}{5^y} \\ &= \frac{1}{29} \cdot \frac{1}{14} \cdot \frac{1}{4} \\ &= \frac{1}{1624} \end{aligned}$$

3. Let $F(x)$ be an k-degree polynomial with integer coefficients such that $F(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_k x^k$ and $0 \le c_i \le 3 \ \ \forall \ \ i \in [0, 3] \cap \mathbb{Z}$. Given that $F(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $F(2)$.

▼ Solution :

$F(\sqrt{3}) = (c_0 + 3c_2 + 3^2 c_4 + \cdots) + (c_1 + 3c_3 + 3^2 c_5 + \cdots)\sqrt{3}$

Therefore, we have that

$$(c_0 + 3c_2 + 3^2 c_4 + \cdots) = 20 \text{ and } (c_1 + 3c_3 + 3^2 c_5 + \cdots) = 17$$

This corresponds to the base-3 expansions of 20 and 17. This gives us that $F(x) = 2 + 2x + 2x^3 + 2x^4 + x^5$. So $F(2) = 86.$

Note : $F(2) = 82$ is also accepted.